Lotka-Volterra Predator Prey Model

MAT 1110

Lab Goals

- Understand a simple predator-prey system
- Become familiar with the Lotka-Volterra model
- Find and understand equilibrium for the given populations
- Be able to work efficiently with the Berkeley Madonna software
- Expand the Lotka-Volterra model

Introduction

We have, so far in our discussion of differential equations, concentrated on modeling only one population. In your homework you analyzed exponential and logistic models by finding their equilibrium solutions and the stability of those solutions. It was assumed during the construction of these models that such populations were not acted on significantly by any other organism that could be present in the environment. While single-population models are applicable in many situations, there are instances in which the interaction between two organisms plays a significant role in the dynamics of their respective populations and should be investigated. One such interaction occurs between predators and prey. A predator-prey system occurs when one organism, the prey, serves as a resource (i.e. food) for another organism, the predator. This relationship has a beneficial effect on the predator's population; as predators obtain more food, they are able to produce more predators. Similarly, there is a detrimental effect on the prey's population since the predator eliminates them.

Modeling Predator and Prey Populations

One of the simplest models used to investigate predator-prey population dynamics is the Lotka-Volterra model. While it is not usually accurate for more intricate systems, it is often used as a basis for more complex models. The Lotka-Volterra model is a system of two differential equations: one that describes the prey's population, and one that describes the predator's population. There are a few critical assumptions, mainly that:

- The prey has unlimited resources.
- The prey's only threat is the predator.
- The predator is a specialist; i.e., the predator's only food supply is the prey.
- The predator's growth depends on the prey it catches.

These assumptions lead straight to the parameters for the model:
a - growth rate of the prey
b - efficiency of the predator's ability to capture prey
c - death rate of the predator
d - growth rate of the predator

The efficiency rate b ranges between 0 and 1. For the predator's capture efficiency b, a value close to zero indicates the predator routinely cannot capture the prey when an encounter occurs. Likewise a value of 1 for b indicates that when an encounter occurs the predator always captures the prey. Note also that the growth rate of the predator depends on both populations, while the growth rate of the prey depends only on the prey's population.

Lastly, with these parameters defined, the differential equations are (using H to represent prey and L to represent predators to be consistent with later parts of the lab):

$$\frac{dH}{dt} = aH - bHL$$
$$\frac{dL}{dt} = -cL + dHL$$

These equations represent the change in the prey and predator populations as time passes. Note that $\frac{dH}{dt}$ has an L term, and that $\frac{dL}{dt}$ likewise has an H term. This is because the two populations are interconnected; the size of one population has an effect on the size of the other population. The quantity HL indicates the amount the two organisms encounter each other. When either population is low the encounter rate is low. When the populations are high so is the encounter rate.

Canadian Lynx and Snowshoe Hare

Two organisms whose populations can be reasonably modeled with the Lotka-Volterra model are the Canadian lynx and the snowshoe hare. Data has been collected on these populations for almost a century through the pelt trading records of the Hudson Bay Company. Using this data together with the Lotka-Volterra model, the populations for both animals can be accounted for largely due to their predator-prey relationship. To view a graph of the actual data and more of the history behind predator-prey dynamics, visit the following web-site:
http://www.math.duke.edu/education/ccp/materials/diffeq/predprey/pred1.html

Building the model

1. Using the following values for the model's parameters, input the two differential equations into Berkeley Madonna. (You may use a flowchart to help you build the model or you may simply input the equations into the equation window.)

   a = .03
   b = .2
   c = .4
   d = .01

2. Change STOPTIME to 1000. Run the model with an initial population of 20 (in thousands) for both the predator and prey.

   init H = 20
   init L = 20

   Question 1: Using the given parameters, what is the *Canadian lynx* population at time t=461?

   Question 2: Using the given parameters, what is the *snowshoe hare* population at time t=461?

   Question 3: Using the given parameters, what is the approximate maximum population of hares? of lynx?

3. Define a slider for the parameter b with the options linear, minimum 0, maximum 1, and increment .05. Click the right arrow on the slider and watch the effect changing this parameter has on the populations.

   Question 4: Describe what happens to the populations as b approaches 1 in terms of the interaction between the two animals.

   Question 5: Solve for the equilibria of each of the two differential equations in terms of the constants (a, b, c, d).

   When finished, remove the slider definition and reset the parameters using the Parameter Window. Close any runs you have open.

   Question 6: Why can we not use the Stability Theorem (Theorem 5.1) to determine the stability of the equilibrium solutions of this model?

Expanding the model

As said above, the Lotka-Volterra model is the simplest way to model predator-prey interaction. Researchers have constantly been expanding this model in order to incorporate more realistic situations. One of the main problems with the Lotka-Volterra model occurs when no prey exist in the system.

Question 7: According to the Lotka-Volterra model, what happens when there are no predators in the system, i.e., when initial predator population is 0? What is unrealistic about this behavior?

To remedy the above problem, it has been suggested to incorporate a carrying capacity term into the model. A new constant M added to the model will allow for the situation when no predators are present. The new differential equations, incorporating the constant M are as follows:

$$\frac{dH}{dt} = a\left(1 - \frac{H}{M}\right)H - bHL$$

$$\frac{dL}{dt} = -cL + dHL$$

Carrying Capacity

1. In Berkeley Madonna, modify your existing equation window (or flowchart) to reflect the new formulas. Do not forget to add a line for M. Set M=30. Reset your values for a, b, c, and d to their original values.
2. Define a slider for H (min 0, max 60), L (min 0, max 60), a (min 0, max 2), b (min 0, max 1), c (min 0, max 2), d (min 0, max 2), and M (min 0, max 100).

   Question 8: Experiment with different values for the seven variables and see if you can discover a combination in which both species will live in equilibrium. You may need to expand the Stoptime in order to see what will happen over a longer period of time.

   Question 9: Solve for the equilibria of the two new equations (with carrying capacity) describing the predator-prey interaction. (Remember you are looking for values H^* such that $f(H^*)=0$.) Do your solutions make sense based on the experiment of question 8? Give specific values for a, b, c, etc. that enable the two species to coexist. What equilibrium values do the hares and lynxes reach based on your choices of a, b, c, etc.?

Conclusion

Adding a carrying capacity term is just one way to modify the predator-prey interaction. In reality, there are many other things that need to be taken into account. Yet, the Lotka-Volterra model gives an intriguing example to begin studies of predator-prey dynamics. Researchers are constantly seeking new and better ways to modify the differential equations to more accurately fit the activity that occurs.

Question 10: Give an example of another fact or issue that might need to be incorporated into the Lotka-Volterra model in order for it to lose some of the restrictions placed on it. Refer back to the criteria for employing the Lotka-Volterra model. How could these issues be addressed?

Questions

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